Theme Round Solutions

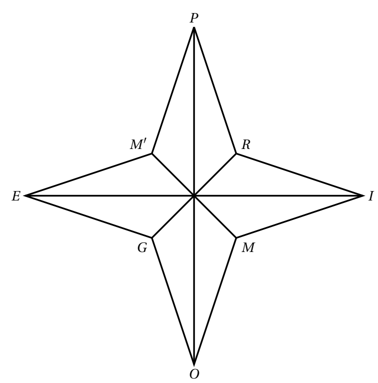
LMT Fall 2024

December 14, 2024

Gaming

At math team, the main activity is math but a concerning close second is gaming. Here are some of our favorite games.

1. **[6]** In Genshin Impact, $PRIMOGEM'$ is the octagon in the diagram below. Let A be the intersection of PO and IE. Suppose $PR = RI = IM = MO = OG = GE = EM' = M'P$, $AP = AI = AO = AE = 4$, and $AR = AM = AG = AM' = \sqrt{2}$. Find the area of $PRIMOGEM'$.



Proposed by: Ella Kim

Solution. $\boxed{16}$

$PRIMOGEM'$ is made up of the square $RMGM'$ with side length 2 and 4 triangles congruent to PRM', which has base 2 and height 3. Therefore, the area of $PRIMOGEM'$ is $2^2 + 4 \cdot \frac{1}{2} \cdot 2 \cdot 3 = \boxed{16}$. □

2. **[8]** In Pokemon, there are 10 indistinguishable Poke Beans in a pile. Pikachu eats a prime number of Poke Beans. Charmander eats an even number of Poke Beans. Snorlax eats an odd number of Poke Beans. Find the number of ways for the three Pokemon to eat all 10 Poke Beans.

Proposed by: Ella Kim

Solution. $\boxed{9}$

Pikachu must eat an odd number of Poke Beans. We perform casework on the number of Poke Beans Pikachu eats.

Case 1: Pikachu eats 3. In this case, Charmander can eat any even amount of Poke Beans from 0 to 6 inclusive, giving 4 ways for this case.

Case 2: Pikachu eats 5. In this case, Charmander can eat any even amount of Poke Beans from 0 to 4 inclusive, giving 3 ways for this case.

Case 3: Pikachu eats 7. In this case, Charmander can eat any even amount of Poke Beans from 0 to 2 inclusive, giving 2 ways for this case.

Summing all the possibilities, we get a final answer of $2 + 3 + 4 = \boxed{9}$. □

3. **[10]** In Survev.io, Calvin observes that he has exactly twice as much blue ammo as red ammo. After firing one blue bullet and 9 red bullets, he remarks that the amount of blue ammo he has is divisible by 5 and the amount of red ammo he has is divisible by 7. Find the least amount of red ammo he could have started with.

   Proposed by: Evin Liang

   Solution. $\boxed{23}$

   Let x be the starting amount of red ammo. Then the starting amount of blue ammo is $2x$. So we get $2x-1 \equiv 0 \pmod 5$ and $x-9 \equiv 0 \pmod 7$. Thus $x \equiv 23 \pmod{35}$. So the least possible value is $x = \boxed{23}$. □

4. **[12]** In Brawl Stars, Rico can shoot his opponent directly or make his bullet bounce off the wall at the same angle. His opponent is 15 feet in front of him and there are infinitely long walls 1 feet to the left and right of Rico. If Rico's bullet travels d feet before hitting the opponent, find the sum of all possible integer values of d.

   Proposed by: Jacob Xu

   Solution. $\boxed{209}$

   Every time Rico's bullet bounces reflect the whole system across the side it bounces off of. In this way the path becomes a straight line, and we get a right triangle with legs equal to 15, and the number of bounces. Thus we must find all Pythagorean Triples with 15 as a leg. This is the sum of all positive integers a for which there is some b satisfying $a^2 - b^2 = 15^2$.

   If two factors $(a+b)$ and $(a-b)$ multiply to 15^2 then a is half their sum. The possible products are

   $$1 \cdot 225, 3 \cdot 75, 5 \cdot 45, 15 \cdot 15.$$

   Thus when we sum this over all pairs which multiply to 15^2 we get all divisors of 225 with 15 counted twice. This is then $\frac{\sigma(15^2)+15}{2} = \frac{(1+3+9)(1+5+25)+15}{2} = \boxed{209}$. □

5. **[14]** In Ace Attorney, Phoenix Wright is rolling a standard fair 20-sided die. He can roll this die up to three times. After each roll, Phoenix can yell "Objection!" to roll again, or "Hold It!" to stop and keep his current number.

   If Phoenix plays optimally to maximize his final number, find the expected value of this number.

   Proposed by: Jacob Xu

   Solution. $\boxed{\frac{72}{5}}$

   We start by considering the second roll: Phoenix has an expected value of 10.5 for the last roll, so he should reroll if he gets between a 1 and a 10. This leads to an expected probability of $\frac{1}{2} \cdot \frac{31}{2} + \frac{1}{2} \cdot \frac{21}{2} = 13$.

   Then, for the first roll, he should reroll if he gets a number less than this expected value of 13, meaning that he will reroll if he gets a 1 through 12. Additionally, if he gets a 13, since it is exactly his expected value, he can choose to either reroll or not, with both choices being optimal. Without loss of generality assume he does not reroll on a 13, meaning his expected value is then

   $$\frac{2}{5} \cdot \frac{33}{2} + \frac{3}{5} \cdot 13 = \frac{33}{5} + \frac{39}{5} = \boxed{\frac{72}{5}}.$$

   □

Brainrot

What is up sigmas!? If your attention span has already elapsed feel free to scroll and start the problems. At this point, brainrot is everywhere: it is the Oxford Dictionary's 2024 word of the year, and here at LMT, certain members are known to say things like "sigma sigma on the wall" quite frequently.

1. **[6]** Suppose h, i, o are real numbers that satisfy the products $hi = 12$, $ooh = 18$, and $hohoho = 27$. Find the value of the product $ohio$.

   Proposed by: Muztaba Syed

Solution. $\boxed{432}$

The third equation tells us $oh = 3$. From the second equation $o = 6$. Then $ohio = hi \cdot o^2 = 12 \cdot 6^2 = \boxed{432}$. □

2. **[8]** A positive n is called *sigma rizz* if the sum of its digits is equal to two times the number of digits it has. Find the number of sigma rizz numbers less than 1000.

Proposed by: Benjamin Yin

Solution. $\boxed{26}$

Case 1: 1 Digit. The sum of the digits must be 2, so there is only 1 solution in this case.

Case 2: 2 Digits. The sum of the digits must be 4, and the first digit must be at least 1. The solutions are 13, 22, 31, 40, so there are 4 solutions for this case.

Case 3: 3 Digits. The sum of the digits must be 6, and the first digit must be at least 1. By Stars and Bars, there are $\binom{7}{2} = 21$ solutions for this case.

Thus, the final answer is $1 + 4 + 21 = \boxed{26}$. □

3. **[10]** Let MEW and MOG be isosceles right triangles such that E, M, O are collinear in that order and G, M, W are collinear in that order. Suppose $ME = MW = \sqrt{6-4\sqrt{2}}$ and $MO = MG = \sqrt{6+2\sqrt{2}}$. Find the least possible area of a circle which contains both triangles MOG and MEW.

Proposed by: Muztaba Syed

Solution. $\boxed{(6-\sqrt{2})\pi}$

This circle is the circumcircle of $OWEG$, so its center X lies on the common perpendicular bisector of WE and OG. Replace $\sqrt{6-4\sqrt{2}}$ and $\sqrt{6+3\sqrt{2}}$ with $a\sqrt{2}$ and $b\sqrt{2}$.

Note that $\angle GXE = 2\angle GWE = 2 \cdot 45^\circ = 90^\circ$. Thus $GE = r\sqrt{2}$ where r is the radius of the circle. But by the Pythagorean Theorem on GME we see $GE^2 = 2(a^2 + b^2) = 12 - 2\sqrt{2}$. This means $r^2 = 6 - \sqrt{2}$, and the area is $(6-\sqrt{2})\pi$. □

4. **[12]** Let S, K, I, B, D, Y be distinct integers from 0 to 9, inclusive. Given that they follow this equation:

$$\begin{array}{rcccc} & S & K & I & B \\ - & I & D & I & D \\ \hline & & & D & Y \end{array}$$

find the maximum value of $\overline{SKIBIDI}$.

Proposed by: Jonathan Liu and Ryan Tang

Solution. $\boxed{8075797}$

First note that the third column forces D to be either 0 or 9 since there can either be 0 or 1 carry. If $D = 0$, then the 2nd column would have no carry, which is impossible since S and I are distinct. Thus, $D = 9$. Thus, we have the following system:

$$\begin{array}{rcccc} & S & K & I & B \\ - & I & 9 & I & 9 \\ \hline & & & 9 & Y \end{array}$$

Now, since there must be a carry in the second column, we have that $K < 9$. Thus, $K + 10 - 9 = 1, 0$, which forces $K = 0$.

$$\begin{array}{rcccc} & S & 0 & I & B \\ - & I & 9 & I & 9 \\ \hline & & & 9 & Y \end{array}$$

From here, we just want to maximize $SKIBIDI$, and so we start by maximizing S. Clearly, $S = 8$ is maximum, and this is achievable when $I = S - 1 = 7$.

$$\begin{array}{rcccc} & 8 & 0 & 7 & B \\ - & 7 & 9 & 7 & 9 \\ \hline & & & 9 & Y \end{array}$$

Finally, we also must have that $B+10-9=Y$. Using the digits that remain, it follows that $B=5, Y=6$. Thus, we get the equation:

$$\begin{array}{rcccc} & 8 & 0 & 7 & 5 \\ - & 7 & 9 & 7 & 9 \\ \hline & & & 9 & 6 \end{array}$$

which indeed works. □

5. **[14]** Tnag is repeating the phrase "sigma sigma on the wall" an infinite number times. Between each word, there is exactly one second of pause. Adam has heard the phrase so many times that he has come up with a game using two numbers x and y: Start with a score of 0.

- At a random time, Adam will hear the word a (each of the 5 words are equally likely to be heard).
- Then
  - if a is "sigma", Adam will multiply his score by x, and
  - if a is any of the other words, Adam will add y to his score.

Let $f(x,y)$ be Adam's expected score after infinitely many steps. Find

$$\sum_{n=2}^{\infty} f\left(\frac{1}{n}, \frac{1}{n^2}\right).$$

Proposed by: Adam Ge and Ryan Tang

Solution. $\boxed{\frac{3}{2}}$

I claim that $f(x,y)=\frac{3y}{2-2x}$.

Let E_n be the expected score after n steps. We have that $E_n = \frac{2}{5}\cdot(xE_{n-1})+\frac{3}{5}\cdot(E_{n-1}+y) = \left(\frac{2x}{5}+\frac{3}{5}\right)E_n + \frac{3y}{5}$ from Linearity of Expectation.

Let S be the generating function of E_n, i.e. $S=\sum_{n\ge 0} E_n \cdot z^n$. Now, by our recurrence relation, $\frac{S}{z} = \left(\frac{2x}{5}+\frac{3}{5}\right)S + \sum_{n\ge 0}\frac{3y}{5}z^n = \left(\frac{2x}{5}+\frac{3}{5}\right)S + \frac{\frac{3y}{5}}{1-z}$. Solving for S, we get

$$\begin{aligned} S &= \frac{\frac{\frac{3y}{5}}{1-z}}{\frac{1}{z}-\frac{2x}{5}-\frac{3}{5}} \\ &= \frac{3yz}{(1-z)(1-\frac{2x+3}{5}z)} \\ &= \frac{3yz}{2-2x}\left(\frac{1}{1-z} - \frac{1}{1-\frac{2x+3}{5}z}\right) \\ &= \frac{3y}{2-2x}\sum_{n\ge 1}\left(1-\left(\frac{2x+3}{5}\right)^n\right)z^n \end{aligned}$$

Since we know that $x<1$, we have that the limit would be $\frac{3y}{2-2x}$.

Alternatively (as well as unrigorously), we can assume convergence. Hence, the expected score is E. Hence, we have that $E=\left(\frac{2x}{5}+\frac{3}{5}\right)E+\frac{3y}{5}$, and thus, $E=\frac{3y}{2-2x}=f(x,y)$. □

Hence, $f(\frac{1}{x},\frac{1}{x^2}) = \frac{3}{2(x-1)x}$. Taking the sum, from telescoping, $\boxed{\frac{3}{2}}$ is our answer. □

Rappers

This year was a very eventful one for rap. Let's celebrate by going through what these rappers did in their spare time.

1. **[6]** Travis Scott says "FEIN" every 0.8 seconds. Find the tens digit of the number of times he says "FEIN" in 1 minute.

   Proposed by: William Hua

Solution. $\boxed{7}$

$$\frac{60\text{ sec}}{0.8\frac{\text{sec}}{\text{FEIN}}} = 75\text{ FEINs}.$$

So the answer is $\boxed{7}$. □

2. **[8]** Eminem is trying to find the real Slim Shady in a row of 2025 indistinguishable Slim Shady clones, one of which is the real Slim Shady. Eminem randomly guesses, and if he guesses wrong, a new clone joins the row and all the clones randomly rearrange themselves. He keeps guessing as more identical clones are added, trying to find the real Slim Shady. Find the probability that he will eventually find him within 15 guesses.

   Proposed by: Kalina Liu

   Solution. $\boxed{\frac{15}{2039}}$

   By complementary counting, the probability that the real Slim Shady is found within 15 guesses is equal to one minus the probability that he is not found. That probability is $\frac{2024}{2025}\cdot\frac{2025}{2026}\cdot\ldots\frac{2038}{2039}=\frac{2024}{2039}$, so the complement is then $\boxed{\frac{15}{2039}}$. □

3. **[10]** Kendrick Lamar and Drake are cutting their circular beef to share with their fans. The cuts must pass all the way from one side of the beef to the other, and no other modifications may be performed on the beef (e.g. folding, eating, stacking, etc.). Find the minimum number of cuts they will need to split their beef into 2024 pieces.

   Proposed by: Atticus Oliver

   Solution. $\boxed{64}$

   Notice that the nth cut can intersect any of the previous $n-1$ cuts, thus splitting as many as n regions these cuts had made make in half, therefore creating up to n regions; i.e. for n cuts, there are at most $1+\sum_{i=0}^{n} n$ regions. This implies $2024 \le 1+\sum_{i=0}^{n} n = \frac{n^2+n}{2}+1$. Notice that this can be solved by setting 2024 equal to $\frac{n^2+n}{2}+1$ and then taking the ceiling of this n. Then $2024-1=\frac{n^2+n}{2}$, or $4046 = n(n+1)$. Notice that $63\cdot 64 = 4032$ and $64\cdot 65 = 4160$, which implies that $63 < n < 64$, so there will be at minimum $\boxed{64}$ cuts to split the beef into 2024 pieces. □

4. **[12]** Let NAS be a triangle such that $NA = NS = 5$ and $AS = 6$. Let D be the foot of the altitude from N to AS and E the foot of the altitude from A to NS. Point X lies on line DE outside the triangle such that $XA = \frac{18}{5}$. Find XS.

   Proposed by: Chris Cheng and Samuel Tsui

   Solution. $\boxed{\frac{24}{5}}$

   Note $ND = 4$, $AE = \frac{24}{5}$, $AD = 3$, and $NE = \frac{7}{5}$, thus by Ptolemy's $DE = 3$. This means that $DES \cong DXA$ so $DX = DE = DS = 3$. Therefore $\angle XSE = 90^\circ$ so by Pythagorean theorem $XS = \boxed{\frac{24}{5}}$. □

5. **[14]** Kanye West's favorite positive integer this year is c, and last year it was $c-t = 20011$ (a prime), for some positive integer t relatively prime to c. His two most streamed albums got a and b streams this year and $a-t$ and $b-t$ streams last year with $a > b > c$. Suppose $a \le 1.6\times 10^9$ and his favorite integer in each year divides the number of streams for both albums in the corresponding year. Find the largest possible value of c.

   Proposed by: William Hua

   Solution. $\boxed{39977}$

   Let $a = a'c$ and $b = b'c$. Then, we have $c-t \mid a'c-c, b'c-c$. Since $c-t$ is prime and $c < 2(c-t)$, $c-t \nmid c$, so $c-t \mid a'-1, b'-1$. To minimize a, we let $a'-1 = 2(c-t)$ since $a > b > c$. Then, $a = 2c(c-t)+c = c(2c-2t+1) = 40023c \le 1.6\times 10^9$. Thus, $c \le 39977$.

   A construction for $c = 39977$ would be $t = 19966$ and $a = c(2c-2t+1)$. □

Tiebreaker Estimation

This problem will only be used to break ties for individual aggregate awards. If two competitors are tied the one closest to the answer will win.

1. **[TIEBREAKER]** From the screen, to the ring, to the pen, to the king, KSI has played games, been in brainrot memes, and rapped. To honor him, estimate the number of times the letters K, S, I appear on this round (excluding answer sheet) in that order, potentially with letters in between them.

   Proposed by: Muztaba Syed, James Wu, Danyang Xu

   Solution. $\boxed{496194}$

   To get an estimate one can note that the answer should roughly be $\frac{1}{6}$ times the product of the number of times each of the three letters occurs. Using a rough estimate of their frequencies it is possible to get a reasonable estimate.

   Exact answer verified with code. □